Int. J. Solids Structures Vol. 29, No. 23, pp. 2867–2880, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00

ON THE SYMMETRY CONDITIONS FOR LAMINATED FIBRE-REINFORCED COMPOSITE STRUCTURES

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(Received 3 December 1991; in revised form 29 April 1992)

Abstract—In this paper the use of symmetries in problems concerned with laminated fibre-reinforced composite structures is examined systematically. Incorrect and incomplete treatments of this subject have appeared in the literature and these are clarified herein. A model of a complete structure is not needed provided that the geometry and the loading of the structure show the required symmetry properties, even though the boundary conditions introduced can vary from one symmetry to another. The extent of the reduction in the size of the problem depends upon the laminate layout scheme. Particular comments are made on the proper use of symmetries in geometrically nonlinear problems. Further applications of symmetries in analytical approaches which lead to a reduction in the number of the dimensions of the problem are also discussed.

1. INTRODUCTION

For any body in three-dimensional space, there are three independent types of transformation which reveal particular symmetries (Hammermesh, 1962):

(1) Reflection in a plane (or mirror reflection) denoted as Σ or more specifically as Σ_x if the plane is normal to the x-axis.

(2) Rotation through an angle about some axis, noted as C^n if the angle is $2\pi/n$, or C_x^n if the axis is the x-axis. In particular, C^2 is sometimes termed as reflection in a line, skewsymmetry, inversion symmetry or polar symmetry.

(3) Translation symmetry, denoted by T^Δ if the translation is Δ or by $T_x^{\Delta x}$ if the translation is Δx along the x-axis. Such symmetry may occur only when the body is of infinite extent in the chosen direction.

All other symmetries can be obtained by a combination of these three, e.g. reflection in a point is simply the combination of Σ and C^2.

When symmetries are used in structural mechanics, three factors have to be taken into consideration: geometry, loading and material. The first two were reviewed by Glockner (1973) in great detail and the third was examined by Noor (1976) and Noor and Camin (1976) for laminated anisotropic structures. In spite of the availability of such work, incorrect uses of symmetry conditions for laminated anisotropic structures have appeared in the literature from time to time [see for example Reddy (1984)], mostly in the context of worked examples. The treatments of this question by Reddy (1984) and Mallikarjuna (1991) were devoted specifically to symmetry conditions and were either incorrect or incomplete and, therefore, might mislead. One reason for this is the fact that the symmetry conditions in Noor and Camin's work are in the form of transformations rather than boundary conditions directly which makes them less explicit and therefore they have not received enough attention. There seems to be a need to produce a complete examination of the problem to ensure that symmetry conditions are used correctly, effectively and simply. This paper provides such an explanation in the context of laminated composite structures.

2. SYMMETRIES AND SYMMETRY CONDITIONS

It is helpful first to specify what we mean by "symmetry" and "symmetry conditions". Symmetry is a general physical property which a structural system may possess and symmetry conditions are the conditions implied by this state. We shall come across the word "antisymmetric" used in two different ways. One is antisymmetric loading which results in antisymmetry, i.e. there are reversals in the directions of the variables describing the state of the structure. To emphasize the difference, the terms "symmetric symmetry conditions" and "antisymmetric symmetry conditions" may be used. It is then clear that antisymmetry is a form of symmetry which appears only when the geometry and the material of the structure possess the particular symmetry while the loading changes its sense under the symmetry transformation. The other occasion on which the word is used is in defining the layout scheme of a laminate as an antisymmetric ply. This is commonly accepted terminology and strictly has nothing to do with the symmetry conditions under discussion.

We shall explore all the possible symmetries that the material (fibre-reinforced composite laminates) may possess, regardless of the actual geometry and associated loading. Figure 1 shows a typical material element where x, y and z are local material coordinate axes which do not necessarily coincide with the global axes used to describe the structure. The z-axis is normal to the lamina. For the sake of convenience, the layout of the laminates will be classified into one of three types and the corresponding discussions will be made with orientations referred to the local x-, y-, z-axes. The angles refer to the orientation of the fibres in a particular lamina with respect to the x-axis.

(i) *Cross-ply* (e.g. $[0^\circ/90^\circ/90^\circ]$)

This is a laminate with several layers whose fibres are parallel to or at right angles to the x-axis as indicated. Materials with such a layout possess symmetries Σ_x, Σ_y, C_z^2, T_x and T_y. Furthermore if the layout is middle surface symmetric, e.g. $[0^\circ/90^\circ/90^\circ/0^\circ]$, we have symmetries C_x^2 and C_y^2 while the symmetry Σ_z results from the symmetric layout.

(ii) *Antisymmetric ply* (e.g. $[\alpha_1/\alpha_2/-\alpha_2/-\alpha_1]$)

Materials with this layout possess symmetries C_x^2, C_y^2, C_z^2, T_x and T_y. It should be noted that the so-called antisymmetric cross-ply, e.g. $[0^\circ/90^\circ/0^\circ/90^\circ]$, could be a particular form of this category if the local x- and y-axes are so chosen that they are at 45° to the fibre direction.

(iii) *Arbitrary ply* (e.g. $[\alpha_1/\alpha_1/\alpha_1]$)

Materials of this layout still possess symmetries C_z^2, T_x and T_y in general. It is the symmetry C_z^2 that was not recognized by Reddy (1984) and Mallikarjuna (1991) and therefore it was suggested therein that full structures have to be analysed for such layouts. However for the geometry and the loading possessing the symmetries of the examples in these references, a half structure model is always applicable by using this symmetry.

Each of the symmetries can be expressed in terms of symmetry conditions which can then be applied as boundary conditions to the representative portion of the structure along its edges. Suppose that the geometry and the loading possess the same symmetries as the material. Then the symmetry conditions are given in Table 1, where each material particle on the reference surface is assumed to have six degrees of freedom, translations along the axes u_x, u_y and u_z and rotations about the axes θ_x, θ_y and θ_z. The associated generalized forces are F_x, F_y, F_z, M_x, M_y and M_z. Each of these can be easily identified as sectional

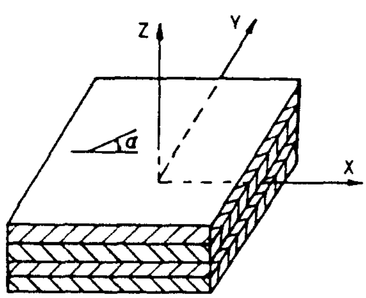

Fig. 1. A typical material element.

Table 1. Symmetry conditions

	Symmetric		Antisymmetric	
	Displacements	Forces	Displacements	Forces
Σ_x				
(for half	$u_x\|_{x=0}=0$	$F_y\|_{x=0}=0$	$u_y\|_{x=0}=0$	$F_x\|_{x=0}=0$
structure	$\theta_y\|_{x=0}=0$	$F_z\|_{x=0}=0$	$u_z\|_{x=0}=0$	$M_y\|_{x=0}=0$
$x \geqslant 0$ or	$\theta_z\|_{x=0}=0$	$M_x\|_{x=0}=0$	$\theta_x\|_{x=0}=0$	$M_z\|_{x=0}=0$
$x \leqslant 0$)				
Σ_y				
(for half	$u_y\|_{y=0}=0$	$F_x\|_{y=0}=0$	$u_x\|_{y=0}=0$	$F_y\|_{y=0}=0$
structure	$\theta_x\|_{y=0}=0$	$F_z\|_{y=0}=0$	$u_z\|_{y=0}=0$	$M_x\|_{y=0}=0$
$x \geqslant 0$ or	$\theta_z\|_{y=0}=0$	$M_y\|_{y=0}=0$	$\theta_y\|_{y=0}=0$	$M_z\|_{y=0}=0$
$x \leqslant 0$)				
C_x^2	$u_y\|_{y=0}=0$			$F_y\|_{y=0}=0$
(for half		$F_x\|_{y=0}=0$	$u_x\|_{y=0}=0$	•
structure	$u_z\|_{y=0}=0$			$F_z\|_{y=0}=0$
$y \geqslant 0$ or	$\theta_y\|_{y=0}=0$			$M_y\|_{y=0}=0$
$y \leqslant 0$)		$M_x\|_{y=0}=0$	$\theta_x\|_{y=0}=0$	
	$\theta_z/_{y=0}=0$			$M_z\|_{y=0}=0$
C_y^2	$u_x\|_{x=0}=0$			$F_x\|_{x=0}=0$
(for half		$F_y\|_{x=0}=0$	$u_y\|_{x=0}=0$	
structure	$u_z\|_{x=0}=0$			$F_z\|_{x=0}=0$
$x \geqslant 0$ or	$\theta_x\|_{x=0}=0$			$M_x\|_{x=0}=0$
$x \leqslant 0$)		$M_y\|_{x=0}=0$	$\theta_y\|_{x=0}=0$	
	$\theta_z\|_{x=0}=0$			$M_z\|_{x=0}=0$
C_z^2	$u_x\|_{x=0,y}=-u_x\|_{x=0,-y}$	$F_x\|_{x=0,y}=F_x\|_{x=0,-y}$	$u_x\|_{x=0,y}=u_x\|_{x=0,-y}$	$F_x\|_{x=0,y}=-F_x\|_{x=0,-y}$
(for half	$u_y\|_{x=0,y}=-u_y\|_{x=0,-y}$	$F_y\|_{x=0,y}=F_y\|_{x=0,-y}$	$u_y\|_{x=0,y}=u_y\|_{x=0,-y}$	$F_y\|_{x=0,y}=-F_y\|_{x=0,-y}$
structure	$u_z\|_{x=0,y}=u_z\|_{x=0,-y}$	$F_z\|_{x=0,y}=-F_z\|_{x=0,-y}$	$u_z\|_{x=0,y}=-u_z\|_{x=0,-y}$	$F_z\|_{x=0,y}=F_z\|_{x=0,-y}$
$x \geqslant 0$ or	$\theta_x\|_{x=0,y}=-\theta_x\|_{x=0,-y}$	$M_x\|_{x=0,y}=M_x\|_{x=0,-y}$	$\theta_x\|_{x=0,y}=\theta_x\|_{x=0,-y}$	$M_x\|_{x=0,y}=-M_x\|_{x=0,-y}$
$x \leqslant 0$)	$\theta_y\|_{x=0,y}=-\theta_y\|_{x=0,-y}$	$M_y\|_{x=0,y}=M_y\|_{x=0,-y}$	$\theta_y\|_{x=0,y}=\theta_y\|_{x=0,-y}$	$M_y\|_{x=0,y}=-M_y\|_{x=0,-y}$
	$\theta_z\|_{x=0,y}=\theta_z\|_{x=0,-y}$	$M_z\|_{x=0,y}=-M_z\|_{x=0,-y}$	$\theta_z\|_{x=0,y}=-\theta_z\|_{x=0,-y}$	$M_z\|_{x=0,y}=M_z\|_{x=0,-y}$
C_z^2	$u_x\|_{x,y=0}=-u_x\|_{-x,y=0}$	$F_x\|_{x,y=0}=F_x\|_{-x,y=0}$	$u_x\|_{x,y=0}=u_x\|_{-x,y=0}$	$F_x\|_{x,y=0}=-F_x\|_{-x,y=0}$
(for half	$u_y\|_{x,y=0}=-u_y\|_{-x,y=0}$	$F_y\|_{x,y=0}=F_y\|_{-x,y=0}$	$u_y\|_{x,y=0}=u_y\|_{-x,y=0}$	$F_y\|_{x,y=0}=-F_y\|_{-x,y=0}$
structure	$u_z\|_{x,y=0}=u_z\|_{-x,y=0}$	$F_z\|_{x,y=0}=-F_z\|_{-x,y=0}$	$u_z\|_{x,y=0}=-u_z\|_{-x,y=0}$	$F_z\|_{x,y=0}=F_z\|_{-x,y=0}$
$y \geqslant 0$ or	$\theta_x\|_{x,y=0}=-\theta_x\|_{-x,y=0}$	$M_x\|_{x,y=0}=M_x\|_{-x,y=0}$	$\theta_x\|_{x,y=0}=\theta_x\|_{-x,y=0}$	$M_x\|_{x,y=0}=-M_x\|_{-x,y=0}$
$y \leqslant 0$)	$\theta_y\|_{x,y=0}=-\theta_y\|_{-x,y=0}$	$M_y\|_{x,y=0}=M_y\|_{-x,y=0}$	$\theta_y\|_{x,y=0}=\theta_y\|_{-x,y=0}$	$M_y\|_{x,y=0}=-M_y\|_{-x,y=0}$
	$\theta_z\|_{x,y=0}=\theta_z\|_{-x,y=0}$	$M_z\|_{x,y=0}=-M_z\|_{-x,y=0}$	$\theta_z\|_{x,y=0}=-\theta_z\|_{-x,y=0}$	$M_z\|_{x,y=0}=M_z\|_{-x,y=0}$
	$u_x\|_{x,y}=u_x\|_{x+\Delta x,y}$	$F_x\|_{x,y}=F_x\|_{x+\Delta x,y}$	$u_x\|_{x,y}=-u_x\|_{x+\Delta x,y}$	$F_x\|_{x,y}=-F_x\|_{x+\Delta x,y}$
	$u_y\|_{x,y}=u_y\|_{x+\Delta x,y}$	$F_y\|_{x,y}=F_y\|_{x+\Delta x,y}$	$u_y\|_{x,y}=-u_y\|_{x+\Delta x,y}$	$F_y\|_{x,y}=-F_y\|_{x+\Delta x,y}$
	$u_z\|_{x,y}=u_z\|_{x+\Delta x,y}$	$F_z\|_{x,y}=F_z\|_{x+\Delta x,y}$	$u_z\|_{x,y}=-u_z\|_{x+\Delta x,y}$	$F_z\|_{x,y}=-F_z\|_{x+\Delta x,y}$
$T_x^{\Delta x}$	$\theta_x\|_{x,y}=\theta_x\|_{x+\Delta x,y}$	$M_x\|_{x,y}=M_x\|_{x+\Delta x,y}$	$\theta_x\|_{x,y}=-\theta_x\|_{x+\Delta x,y}$	$M_x\|_{x,y}=-M_x\|_{x+\Delta x,y}$
	$\theta_y\|_{x,y}=\theta_y\|_{x+\Delta x,y}$	$M_y\|_{x,y}=M_y\|_{x+\Delta x,y}$	$\theta_y\|_{x,y}=-\theta_y\|_{x+\Delta x,y}$	$M_y\|_{x,y}=-M_y\|_{x+\Delta x,y}$
	$\theta_z\|_{x,y}=\theta_z\|_{x+\Delta x,y}$	$M_z\|_{x,y}=M_z\|_{x+\Delta x,y}$	$\theta_z\|_{x,y}=-\theta_z\|_{x+\Delta x,y}$	$M_z\|_{x,y}=-M_z\|_{x+\Delta x,y}$
	$u_x\|_{x,y}=u_x\|_{x,y+\Delta y}$	$F_x\|_{x,y}=F_x\|_{x,y+\Delta y}$	$u_x\|_{x,y}=-u_x\|_{x,y+\Delta y}$	$F_x\|_{x,y}=-F_x\|_{x,y+\Delta y}$
	$u_y\|_{x,y}=u_y\|_{x,y+\Delta y}$	$F_y\|_{x,y}=F_y\|_{x,y+\Delta y}$	$u_y\|_{x,y}=-u_y\|_{x,y+\Delta y}$	$F_y\|_{x,y}=-F_y\|_{x,y+\Delta y}$
$T_y^{\Delta y}$	$u_z\|_{x,y}=u_z\|_{x,y+\Delta y}$	$F_z\|_{x,y}=F_z\|_{x,y+\Delta y}$	$u_z\|_{x,y}=-u_z\|_{x,y+\Delta y}$	$F_z\|_{x,y}=-F_z\|_{x,y+\Delta y}$
	$\theta_x\|_{x,y}=\theta_x\|_{x,y+\Delta y}$	$M_x\|_{x,y}=M_x\|_{x,y+\Delta y}$	$\theta_x\|_{x,y}=-\theta_x\|_{x,y+\Delta y}$	$M_x\|_{x,y}=-M_x\|_{x,y+\Delta y}$
	$\theta_y\|_{x,y}=\theta_y\|_{x,y+\Delta y}$	$M_y\|_{x,y}=M_y\|_{x,y+\Delta y}$	$\theta_y\|_{x,y}=-\theta_y\|_{x,y+\Delta y}$	$M_y\|_{x,y}=-M_y\|_{x,y+\Delta y}$
	$\theta_z\|_{x,y}=\theta_z\|_{x,y+\Delta y}$	$M_z\|_{x,y}=M_z\|_{x,y+\Delta y}$	$\theta_z\|_{x,y}=-\theta_z\|_{x,y+\Delta y}$	$M_z\|_{x,y}=-M_z\|_{x,y+\Delta y}$

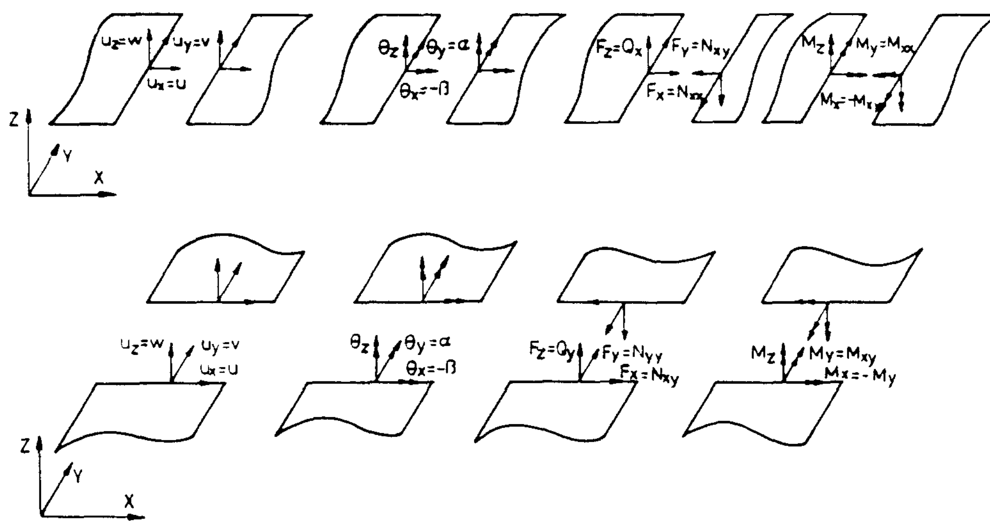

Fig. 2. Identification of generalized displacements and forces for plates and shells.

variables u, v, w, α, β, N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}, Q_x and Q_y in laminated plates and shells as shown in Fig. 2. Depending on the nature of the loading, the symmetry conditions in Table 1 are given for symmetric and antisymmetric cases respectively. For displacement finite element analysis in particular, one does not need to be concerned about the force conditions in Table 1, since they will be included as natural boundary conditions and satisfied automatically by the variational procedure.

In laminated anisotropic structures, the material property is characterized by the well-known A, B and D matrices [see Vinson and Chou (1975)] and there may be coupling between the components of the stress resultants and the generalized strains to different extents depending on the nature of the layout. Two particular types of coupling are those represented by matrix B and those elements in the A, B and D matrices with subscripts 16 and 26 resulting from the presence of off-axis plies. Reddy (1984) attributed the invalidity of Σ-type symmetry to the former type of coupling. This is not correct. What violates the Σ-type symmetry is the latter type of coupling. What have been violated due to the presence of nonzero elements in the B matrix (specifically, elements B_{11}, B_{12}, B_{22}, B_{66}) are symmetries C_x^2 and C_y^2. This did not seem to be realized by Reddy (1984). Fortunately, none of these types of coupling violates the symmetry C_z^2.

The symmetry conditions given by Mallikarjuna (1991) relate to some particular cases which will be considered later in the examples. However the conclusion that a full structure model has to be used for an arbitrary ply is not valid in view of the symmetry C_z^2 which is satisfied by the loading as well as the geometry for the particular example quoted.

It should be noted that once a structure shows symmetry C_z^2, it makes no difference which plane is chosen to cut the structure into two symmetric or antisymmetric halves as long as the plane passes through the rotation axis. Noor and Camin (1976) preferred a diagonal plane which is obviously not the best choice when quadrilateral elements are used. In this case one has to resort to the concept of dependent nodes to apply such symmetry conditions, which may blur the clarity of the application of such symmetry conditions in certain cases. However for problems in which the only symmetry shown in the structure, usually of type Σ_x, Σ_y, C_x^2 or C_y^2, is between the two halves cut by such diagonal planes, the concept of dependent nodes can be very useful.

To show how the symmetry conditions are applied in practice, it is helpful to look at a few examples. Three structures are given in Fig. 3. When the layouts of these structures are cross-ply, the symmetry conditions are no different from those for structures made from isotropic materials which are familiar to the reader. However, it may be worth noting that the loading could be antisymmetric as in Fig. 3(b) for Σ_x symmetry. Only the symmetric case such as Fig. 3(a) was given for this layout by Mallikarjuna (1991). Herein we shall concentrate on layouts other than cross-ply.

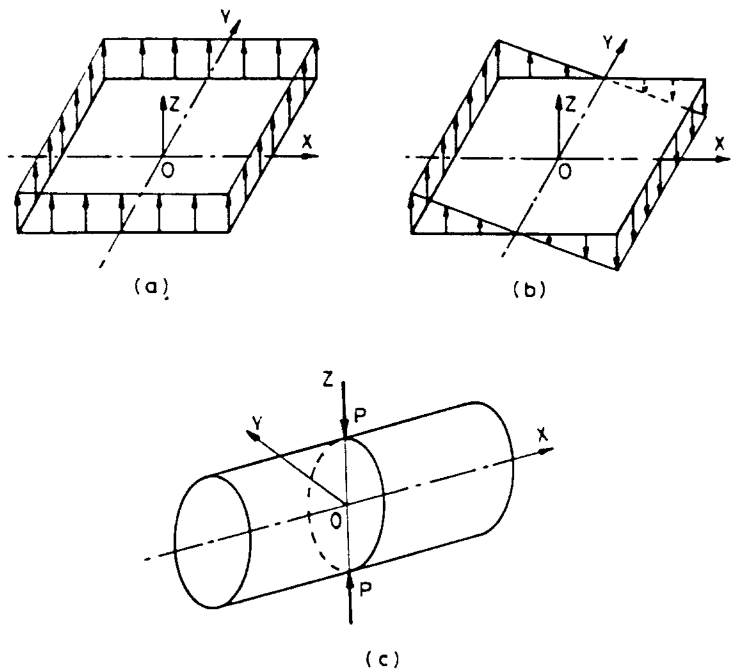

Fig. 3. Examples: (a) A rectangular plate under uniform pressure. (b) A rectangular plate under linearly distributed pressure. (c) A cylinder under a pair of concentrated loads.

For an antisymmetric ply, e.g. $[\alpha_1/\alpha_2/-\alpha_2/-\alpha_1]$, the structure in Fig. 3(a) possesses C_x^2 and C_y^2 symmetries (C_z^2 as well, but it is not independent, actually it is a simple combination of C_x^2 and C_y^2) while the loading is antisymmetric under both transformations. (The loading is symmetric under Σ_x and Σ_y, but the material does not possess such symmetries.) If the plate is cut along the x- and y-axes and modelled by a quarter of it, the boundary conditions are

$$\left.\begin{aligned} u = \beta = 0, &\quad \text{at } y = 0 \quad \text{(antisymmetric)} \\ \text{and} \qquad & \\ v = \alpha = 0, &\quad \text{at } x = 0 \quad \text{(antisymmetric)} \end{aligned}\right\}. \qquad (1)$$

These are exactly what were proposed by Mallikarjuna (1991). However, the situation is quite different for the structure in Fig. 3(b) where the loading is antisymmetric [as in Fig. 3(a)] under C_x^2 but symmetric under C_y^2. (The loading is antisymmetric under Σ_y but again the material does not possess such symmetry.) Therefore the boundary conditions for the quarter plate are

$$\left.\begin{aligned} u = \beta = 0, &\quad \text{at } y = 0 \quad \text{(antisymmetric)} \\ \text{and} \qquad & \\ u = w = \beta = 0, &\quad \text{at } x = 0 \quad \text{(symmetric)} \end{aligned}\right\}. \qquad (2)$$

It is seen that a plate with antisymmetric plies can be exactly represented by a quarter model provided that the geometry possesses the same symmetries C_x^2 and C_y^2 as does the material, and the loading is either symmetric or antisymmetric under these symmetry transformations. Boundary conditions for the quarter model can be obtained from the symmetry conditions (Table 1) corresponding to the symmetric or antisymmetric nature of the loading.

Moving on to plates with arbitrary ply orientations as in Figs 3(a) and (b), the only symmetry is C_z^2. Thus they can both be modelled by half models. Under this symmetry, the loading is symmetric in Fig. 3(a) and antisymmetric in Fig. 3(b). Suppose the plates are cut along the x-axis, the boundary conditions along it are

$$\left.\begin{aligned} u|_{x,y=0} &= -u|_{-x,y=0} \\ v|_{x,y=0} &= -v|_{-x,y=0} \\ w|_{x,y=0} &= w|_{-x,y=0} \\ \alpha|_{x,y=0} &= -\alpha|_{-x,y=0} \\ \beta|_{x,y=0} &= -\beta|_{-x,y=0} \end{aligned}\right\}. \tag{3}$$

in particular

$$u = v = \alpha = \beta = 0, \quad \text{with } w \text{ free} \quad \text{at } x = y = 0, \tag{4}$$

and

$$\left.\begin{aligned} u|_{x,y=0} &= u|_{-x,y=0} \\ v|_{x,y=0} &= v|_{-x,y=0} \\ w|_{x,y=0} &= -w|_{-x,y=0} \\ \alpha|_{x,y=0} &= \alpha|_{-x,y=0} \\ \beta|_{x,y=0} &= \beta|_{-x,y=0} \end{aligned}\right\}. \tag{5}$$

with

$$w = 0 \quad \text{with} \quad u, v, \alpha, \beta \text{ free} \quad \text{at } x = y = 0 \tag{6}$$

for the plates in Figs 3(a) and (b) respectively.

Of course, to apply the C_2^2 one can cut the plates along any other axis through the origin in the x–y plane such as the diagonals used in Noor *et al.*'s papers and arrive at a similar set of conditions, but such arrangements are obviously not the best choice when quadrilateral elements are used. This is because, in this case, one has to either introduce a very irregular mesh or resort to the concept of dependent nodes [see Noor (1976)] to apply the symmetry in the form of a transformation which is much less explicit than in the form of boundary conditions given above. It should be pointed out that, whatever choice is made, none of the other possibilities can be used at the same time since there is only one independent symmetry. In other words laminates of arbitrary ply can be modelled by half models but never smaller than half.

It is important to notice the difference between boundary conditions (1) and (2) and those given in (3) and (5). The first are in the form of prescribed values (zero in this case) which is quite standard while the latter are in the form of equations relating unknowns. A brief demonstration is provided below of the application of such conditions. Assume that the finite element equilibrium equation is in the form:

$$\begin{bmatrix} \cdots\cdots & & & & & & \\ K_{i,i-1} & K_{i,i} & & & & \text{symm.} & \\ K_{i+1,i-1} & K_{i+1,i} & K_{i+1,i+1} & & & & \\ \cdots\cdots & & & & & & \\ K_{j-1,i-1} & K_{j-1,i} & K_{j-1,i+1} & \cdots & K_{j-1,j-1} & & \\ K_{j,i-1} & K_{j,i} & K_{j,i+1} & \cdots & K_{j,j-1} & K_{j,j} & \\ K_{j+1,i-1} & K_{j+1,i} & K_{j+1,i+1} & \cdots & K_{j+1,j-1} & K_{j+1,j} & K_{j+1,j+1} \\ \cdots\cdots & & & & & & \end{bmatrix} \begin{Bmatrix} \cdot \\ u_i \\ u_{i+1} \\ \cdot \\ u_{j-1} \\ u_j \\ u_{j+1} \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \cdot \\ F_i \\ F_{i+1} \\ \cdot \\ F_{j-1} \\ F_j \\ F_{j+1} \\ \cdot \end{Bmatrix}. \tag{7}$$

Imposing the condition

$$u_i = \pm u_j \tag{8}$$

leads to the following form of eqn (7):

$$\begin{bmatrix} \cdots\cdots\cdots\cdots\cdots & & & & & \\ K_{i,i-1} \pm K_{j,i-1} & K_{i,i} \pm K_{j,j} \pm 2K_{i,j} & & \text{symm.} & & \\ K_{i+1,i-1} & K_{i+1,i} \pm K_{i+1,j} & K_{i+1,i+1} & & & \\ \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots & & & \\ K_{j-1,i-1} & K_{j-1,i} \pm K_{j-1,j} & K_{j-1,i+1} & \cdots & K_{j-1,j-1} & \\ K_{j+1,i-1} & K_{j+1,i} \pm K_{j+1,j} & K_{j+1,i+1} & \cdots & K_{j+1,j-1} & K_{j+1,j+1} \\ \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \end{bmatrix} \begin{Bmatrix} \cdot \\ u_i \\ u_{i+1} \\ \cdot \\ u_{j-1} \\ u_{j+1} \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \cdot \\ F_i \pm F_j \\ F_{i+1} \\ \cdot \\ F_{j-1} \\ F_{j+1} \\ \cdot \end{Bmatrix}. \tag{9}$$

The symmetric property of the stiffness matrix is retained after imposing the conditions but the band width of the particular columns [ith in eqn (9)] can be affected (increased) to some extent. Even so the solution of such a system is always more economical than dealing with a complete structure.

The increases in the band width of particular columns in the stiffness matrix to different extents for different choices of cutting planes could make a slight difference in computing costs, but these differences are usually negligible.

The discussion above shows that a plate with arbitrary ply orientations can be represented exactly by a half model. This contradicts the conclusions drawn by both Reddy (1984) and Mallikarjuna (1991).

The third example in Fig. 3(c) is a cylinder. The main discussion will centre on the case of an arbitrary ply but brief comments will also be made about the antisymmetric ply case. It is obvious that the problem (geometry, loading and material) possesses three symmetries C_x^2, C_y^2 and C_z^2 (x, y, z are the global axes of the structure and are different from those of the material). However, only two of them are independent. Therefore, the structure can be represented by a quarter model. For the quarter $y \geqslant 0$, $z \geqslant 0$, the boundary conditions along the two edges are

from C_z^2:

$$\left.\begin{aligned} u|_{x,y=0,z=R} &= -u|_{-x,y=0,z=R} \\ v|_{x,y=0,z=R} &= -v|_{-x,y=0,z=R} \\ w|_{x,y=0,z=R} &= w|_{-x,y=0,z=R} \\ \alpha|_{x,y=0,z=R} &= -\alpha|_{-x,y=0,z=R} \\ \beta|_{x,y=0,z=R} &= -\beta|_{-x,y=0,z=R} \end{aligned}\right\}, \tag{10}$$

$$u = v = \alpha = \beta = 0, \quad \text{at } x = 0, \quad y = 0, \quad z = R; \tag{11}$$

and from C_y^2:

$$\left.\begin{aligned} u|_{x,y=R,z=0} &= -u|_{-x,y=R,z=0} \\ v|_{x,y=R,z=0} &= -v|_{-x,y=R,z=0} \\ w|_{x,y=R,z=0} &= w|_{-x,y=R,z=0} \\ \alpha|_{x,y=R,z=0} &= -\alpha|_{-x,y=R,z=0} \\ \beta|_{x,y=R,z=0} &= -\beta|_{-x,y=R,z=0} \end{aligned}\right\}, \tag{12}$$

$$u = v = \alpha = \beta = 0, \quad \text{at } x = 0, \quad y = R, \quad z = 0. \tag{13}$$

Here u and v are the in-plane axial and tangential displacements, α and β are the corresponding rotations and w is the lateral deflection. Here, all the conditions correspond to symmetric loading. In practice there could be a case of antisymmetric loading. The boundary conditions could be obtained in a corresponding manner which is quite straightforward. Also, one may prefer to choose other quarters than the one chosen here. The boundary conditions could be obtained in a similar manner to those above.

It is interesting to note that in this case having an antisymmetric ply in the shell structure provides no extra help at all. This is because the local axes of material symmetries C_x^2 and C_y^2 (where here x and y are local axes) are not coincident with the global axes of the structure. This could be regarded as another difference between plates and shells from the viewpoint of symmetry properties.

The above discussions are basically focused on rotation symmetries. Before completing this section, it might be helpful to comment on the two most frequently encountered symmetries associated with reflection and rotation in laminated structures. It has been demonstrated through the examples considered that the rotation symmetries can be expressed in terms of boundary conditions which are of no substantial difference in form from those associated with reflection symmetries. Therefore, once a rotation symmetry is identified, it can be easily interpreted based on the above discussions and the procedure to apply such a boundary condition is familiar to structural analysts. It has also been shown that the rotation symmetries are as effective as reflection symmetries in reducing the size of the model to be analysed. Beyond this, it should be noted that rotation symmetries appear much more frequently in laminated structures than reflection symmetries. One of the major intentions of this paper is to draw the attention of structural analysts to this type of symmetry in laminated structures and to encourage them to make full use of it in practice.

Finally, it might be helpful to recall two well-known facts. Firstly, any arbitrarily distributed load can be decomposed into a symmetric component and an antisymmetric one. This may be useful sometimes since it may be more economical to carry out two separate half structure analyses (one symmetric and one antisymmetric) than to analyse a single, complete structure. Secondly, in some analyses there might be situations where load or load increment appears to be in a neutral state such as in problems of free vibration (Noor and Camin, 1976) and buckling (Li and Reid, 1990). The behaviour of the structure could be either symmetric or antisymmetric and therefore both have to be analysed before one can decide which leads to the lowest frequency or buckling load.

3. CHOOSING THE SYMMETRIES IN GEOMETRICALLY NONLINEAR PROBLEMS

Consider first a simple example of the planar ring shown in Fig. 4. It is obvious that the problem satisfies both symmetric symmetry Σ_y and antisymmetric symmetry Σ_x. As is well known [see for example Niles and Newell (1943)], the former reduces the degree of statical indeterminancy by one whilst the latter produces a reduction of two. The point of this example is to show that a different choice of symmetries does make a difference in the nature of the reduced problem. These differences can be very pronounced since some choices may be unacceptable in certain circumstances when geometric nonlinearity is involved.

As noted above, some problems may possess more than one form of symmetry. To be specific, consider the two simple examples in Fig. 5. The obvious symmetries to choose are the antisymmetries associated with plane reflection. However, there are also rotation symmetries such as C_y^2 for the beam and C_x^2, C_y^2 and C_z^2 for the plate which are mostly

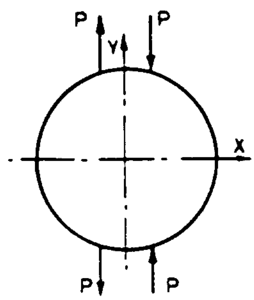

Fig. 4. A planar ring.

symmetric. From the viewpoint of reducing the size of the model of the structure under investigation, both Σ and C^2 have the same result, i.e. the beam is modelled by a half beam and the plate by a quarter plate. However when geometrically nonlinear behaviour is of interest, the difference between choosing one of the two symmetries is very significant, since, as is well known, antisymmetric deformation will violate the symmetry of the geometry of the structure in its deformed configuration and therefore such antisymmetric conditions are unacceptable. This can be seen clearly in the beam where antisymmetry requires that axial force disappears at the centre of the beam, which rules out completely the prime nonlinear effect. On the other hand, if C^2 (C_y^2 for the beam and C_x^2 and C_y^2 for the plate) is used instead of Σ, the structure and load system as well as the deformation are symmetric under the corresponding symmetry transformations, therefore, these are acceptable even for geometrically nonlinear problems.

Thus, it can be proposed that when such nonlinearities are involved, symmetric symmetry conditions must be chosen from among all the existing symmetries rather than antisymmetric ones. More attention should be paid to symmetries associated with C^2 types since it is evident that such symmetries have not received as much attention as those of the Σ type despite the fact that they are more frequently encountered in laminated fibre-reinforced composite structures than those associated with Σ.

4. OTHER APPLICATIONS OF SYMMETRY CONDITIONS

The application of symmetries Σ_x, Σ_y, C_x^2, C_y^2 and C_z^2 to laminated composite structures such as plates and shells is basically straightforward and the outcome is the reduction in the size of the structure under investigation which is especially useful in numerical analyses such as finite element analysis. Translation symmetries $T_x^{\Delta x}$ and $T_y^{\Delta y}$ can fill a similar role for infinitely long plates under periodic load distributions. In this section we explore other applications of symmetry conditions. These again bring simplification and economy by reducing the number of dimensions of the problem. They are therefore very useful both in analytical and numerical analyses although here we shall concentrate only on issues related to analytical studies.

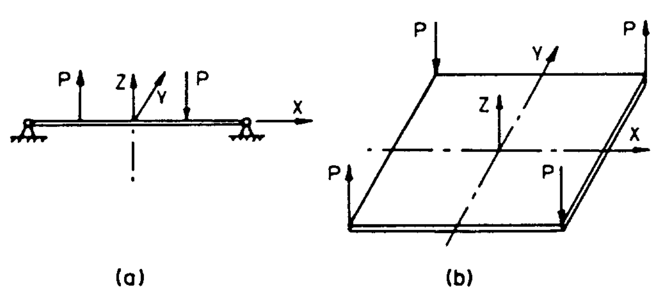

Fig. 5. (a) A simply-supported beam. (b) A rectangular plate under corner loads.

A familiar example is that of plane strain which, as a result of reflection symmetry about any plane in which the problem is defined normal to the longitudinal axis along which the normal strain is identically zero, brings great simplification to the problem since the dimensions are reduced from 3 to 2 in general or for plates and shells from 2 to 1. For fibre-reinforced laminates the plane strain approach only applies to cross-ply structures. The plane reflection type symmetries no longer exist for laminates with any off-axis plies and so no plane strain problems can be posed for such laminates. However, by using other types of symmetry, generalized plane strain problems can be obtained depending on the nature of the load distribution. This is discussed below by assuming that the laminated structure is infinitely long in the y direction and that the geometry and the load are all constant along this direction.

In laminated structures, a three-dimensional analysis is sometimes required in order to provide detailed stress distributions, for example, for laminates with matrix cracks (Hashin, 1985). In most treatments the problem is assumed to be a two-dimensional plane strain problem and therefore the results obtained are restricted to cross-ply laminates. However, by using translation symmetry $T_y^{\Delta y}$ with $\Delta y \to 0$ instead of reflection symmetry Σ_y at any y, such two-dimensional approaches can be extended from cross-ply to any layout including arbitrary ply. This approach results in a generalized plane strain problem for which the governing equations in each lamina are as follows:

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} &= 0 \end{aligned}\right\}, \tag{14}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{13} & 0 & 0 & C_{16} \\ C_{13} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}, \tag{15}$$

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} \\ \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} \end{aligned}\right\}, \tag{16}$$

and

$$\begin{aligned} \varepsilon_y &= 0, \\ \sigma_y &= C_{12}\varepsilon_x + C_{23}\varepsilon_z + C_{26}\gamma_{xy}, \end{aligned} \tag{17}$$

where C_{ij} $(i, j = 1, 2, \ldots, 6)$, is the stiffness matrix of the material (a uniaxially fibre-reinforced composite) with respect to the global axes of the structure. Body forces are not included in the equilibrium equation (14). Boundary conditions are not given since they are standard. As is well known, the interlaminar continuity of stresses and displacements has to be introduced as boundary conditions for each lamina.

By employing plate theory (Vinson and Chou, 1975) the above two-dimensional problem can be further specialized to a one-dimensional problem:

$$\left.\begin{aligned} &\frac{\partial N_{xx}}{\partial x} = 0 \\ &\frac{\partial N_{xy}}{\partial x} = 0 \\ &\frac{\partial M_{xx}}{\partial x} - Q_x = 0 \\ &\frac{\partial M_{xy}}{\partial x} - Q_y = 0 \\ &\frac{\partial Q_x}{\partial x} = p \end{aligned}\right\}, \tag{18}$$

$$\begin{Bmatrix} N_{xx} \\ N_{xy} \\ M_{xx} \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} & 0 & 0 \\ A_{16} & A_{66} & B_{16} & B_{66} & 0 & 0 \\ B_{11} & B_{16} & D_{11} & D_{16} & 0 & 0 \\ B_{16} & B_{66} & D_{16} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{44} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \tag{19}$$

$$\left.\begin{aligned} &\varepsilon_x^0 = \frac{\partial u^0}{\partial x} \\ &\gamma_{xy}^0 = \frac{\partial v^0}{\partial x} \\ &\kappa_x = \frac{\partial \alpha}{\partial x} \\ &\kappa_{xy} = \frac{\partial \beta}{\partial x} \\ &\gamma_{xz} = \alpha + \frac{\partial w}{\partial x} \\ &\gamma_{yz} = \beta \end{aligned}\right\}, \tag{20}$$

and

$$\left.\begin{aligned} &\varepsilon_y^0 = \kappa_y = 0 \\ &N_{yy} = A_{12}\varepsilon_x^0 + A_{26}\gamma_{xy}^0 + B_{12}\kappa_x + B_{26}\kappa_{xy} \\ &M_{yy} = B_{12}\varepsilon_x^0 + B_{26}\gamma_{xy}^0 + D_{12}\kappa_x + D_{26}\kappa_{xy} \end{aligned}\right\}, \tag{21}$$

where p is the lateral distributed load (a function of x only) and superscript 0 refers to the reference plane.

When a laminate is treated as a plate as above, further simplications can be introduced if it is an antisymmetric ply (A_{16}, A_{26}, B_{11}, B_{12}, B_{22}, B_{66}, D_{16} and D_{26} vanish for such layouts) by imposing C_y^2 symmetry at the same time. Attention has to be paid to the loading

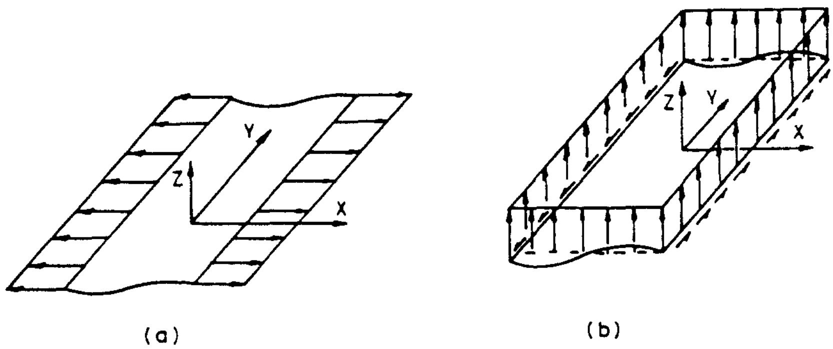

Fig. 6. (a) A long plate under uniform stretching. (b) A long plate under uniform lateral pressure and in-plane shear.

since it could be symmetric or antisymmetric as shown in Fig. 6. For symmetric loading [Fig. 6(a)] the symmetry conditions are

$$\left.\begin{array}{l} v^0 = w = \alpha = 0 \\ N_{xy} = M_{xx} = M_{yy} = Q_x = 0 \end{array}\right\}. \tag{22}$$

eqns (18)–(21) reduce to

$$\left.\begin{array}{l} \dfrac{\partial N_{xx}}{\partial x} = 0 \\ \dfrac{\partial M_{xy}}{\partial x} - Q_y = 0 \end{array}\right\}, \tag{23}$$

$$\begin{Bmatrix} N_{xx} \\ M_{xy} \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{16} & 0 \\ B_{16} & B_{66} & 0 \\ 0 & 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \kappa_{xy} \\ \gamma_{xz} \end{Bmatrix}, \tag{24}$$

$$\begin{Bmatrix} \varepsilon_x^0 \\ \kappa_{xy} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u^0}{\partial x} \\ \dfrac{\partial \beta}{\partial x} \\ \beta \end{Bmatrix}, \tag{25}$$

and

$$\left.\begin{array}{l} \varepsilon_y^0 = \kappa_y = 0 \\ N_{yy} = A_{12}\varepsilon_x^0 + B_{26}\kappa_{xy} \end{array}\right\}. \tag{26}$$

For antisymmetric loading [Fig. 6(b)] we have symmetry conditions

$$\begin{array}{l} u^0 = \beta = 0, \\ N_{xx} = N_{yy} = M_{xy} = Q_y = 0, \end{array} \tag{27}$$

and eqns (18)–(21) reduce to

$$\left.\begin{aligned} \frac{\partial N_{xy}}{\partial x} &= 0 \\ \frac{\partial M_{xx}}{\partial x} - Q_x &= 0 \\ \frac{\partial Q_x}{\partial x} &= p \end{aligned}\right\}, \tag{28}$$

$$\begin{Bmatrix} N_{xy} \\ M_{xx} \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{66} & B_{16} & 0 \\ B_{16} & D_{11} & 0 \\ 0 & 0 & A_{44} \end{bmatrix} \begin{Bmatrix} \gamma^0_{xy} \\ \kappa_x \\ \gamma_{xz} \end{Bmatrix}, \tag{29}$$

$$\left.\begin{aligned} \gamma^0_{xy} &= \frac{\partial v^0}{\partial x} \\ \kappa_x &= \frac{\partial \alpha}{\partial x} \\ \gamma_{xz} &= \alpha + \frac{\partial w}{\partial x} \end{aligned}\right\}, \tag{30}$$

and

$$\left.\begin{aligned} \varepsilon^0_y &= \kappa_y = 0 \\ M_{yy} &= B_{26}\gamma^0_{xy} + D_{12}\kappa_x \end{aligned}\right\}. \tag{31}$$

It should be pointed out that for shells (long cylindrical shells with their axis in the y-direction), whilst translational symmetry is also applicable and a set of simplified shell equations can be obtained analogous to those given in eqns (18)–(21), the simplifications arising from C_x^2 for plates of antisymmetric ply cannot be obtained. The extension of such simplifications to shells of revolution can be obtained in a similar way (Noor and Camin, 1976). The coincidence can be readily seen from the observation that a rotation about the axis of the shell is equivalent to a translation in the circumferential direction, hence the translation symmetry presents itself in the circumferential direction.

All of the applications of symmetries thus far have been to particular structures to reduce the number of degrees of freedom or to simplify the problem. Symmetry can also be used to permit the response of a structure to be deduced from the response of either the same structure under different loading conditions or a different one. If the response is known then, by applying certain symmetry transformations, the original problem can be solved without detailed calculations. This is demonstrated in Fig. 7. If the response of the plate in Fig. 7(a) is known, then all the responses of the plates in Figs 7(b)–(f) can be readily obtained from the corresponding symmetry transformations. However, if the plate is replaced by a curved panel, C_x^2 and C_y^2 [Figs 7(d) and (e)] cannot be used.

5. CONCLUSIONS

Reflection symmetries, for example Σ_x and Σ_y, have been widely used in structural analysis. However, when laminated fibre-reinforced composite structures are concerned, their applications are limited to structures of cross-ply layout. On the other hand, rotation symmetries C_x^2, C_y^2 and C_z^2, which have not received much attention hitherto, can be used for more general layouts and each of them, when applicable, can halve the size of the model to be analysed as does Σ_x or Σ_y. However the boundary conditions introduced by such symmetries are different in form from those of Σ_x and Σ_y. Besides reducing the size of the model to be analysed to the lowest level, identifying all the symmetries available in the structure under investigation can also help in choosing proper symmetries for particular problems, for

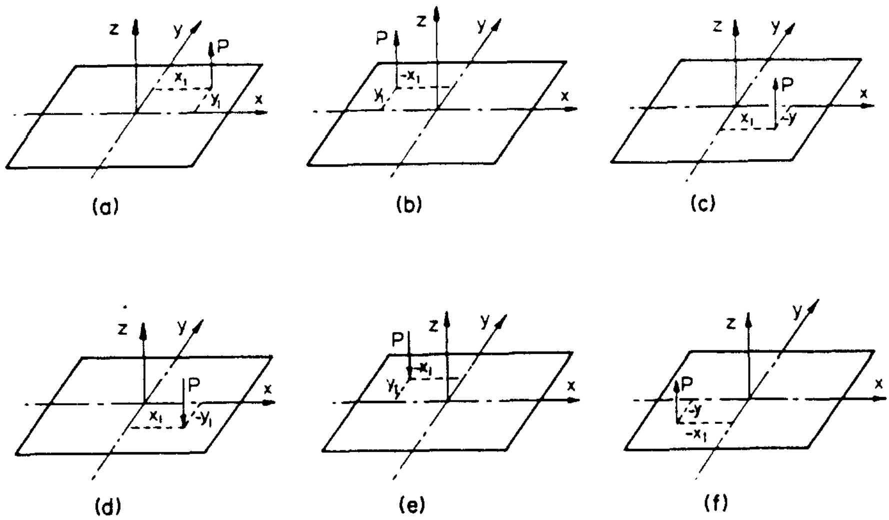

Fig. 7. Equivalent structures: (a) $[x_1/x_2/\ldots/x_n]$. (b) Σ_x: $[-x_1/-x_2/\ldots/-x_n]$. (c) Σ_y: $[-x_1/-x_2/\ldots/-x_n]$. (d) C_x^2: $[x_n/\ldots/x_2/x_1]$. (e) C_y^2: $[x_n/\ldots/x_2/x_1]$. (f) C_z^2: $[x_1/x_2/\ldots/x_n]$.

example symmetric symmetries for geometrically nonlinear problems. Furthermore, translational symmetries can be used to reduce the number of dimensions of some particular problems just as reflection symmetries do, which may be very useful in analytical approaches for such problems as well as in numerical analyses.

Acknowledgements—This work was completed during the course of Ministry of Defence research contract 2044/166/RARDE the support from which is gratefully acknowledged.

The authors wish to express their thanks to Mr P. D. Soden for helpful comments on the original draft of this paper. They would also like to thank Miss C. E. Tyler for typing the manuscript.

REFERENCES

Gloeckner, P. G. (1973). Symmetry in structural mechanics. *Proc. ASCE, J. Struct. Div.* **99**, 71–89.
Hamermesh, H. (1962). *Group Theory and its Application to Physical Problems*. Addison-Wesley, New York.
Hashin, Z. (1985). Analysis of cracked laminates: a variational approach. *Mech. Mater.* **4**, 121–136.
Li, S. and Reid, S. R. (1990). Relationship between the elastic buckling of square tubes and rectangular plates. *J. Appl. Mech.* **57**, 969–973.
Mallikarjuna (1991). An important note on symmetry line boundary conditions in fibre-reinforced laminated anisotropic composites. *Comput. Struct.* **38**, 669–671.
Niles, A. S. and Newell, J. S. (1943). *Airplane Structures* (3rd Edn). Wiley, New York.
Noor, A. K. (1976). Symmetries in laminated composite plates, developments in theoretical and applied mechanics. *Proc. 8th Southeastern Mechanics Conf.* (pp. 225–246).
Noor, A. K. and Camin, R. A. (1976). Symmetry considerations for anisotropic shells. *Int. J. Num. Meth. Engng* **9**, 317–335.
Reddy, J. N. (1984). A note on symmetry considerations in the transient response of unsymmetrically laminated anisotropic plates. *Int. J. Num. Meth. Engng* **20**, 175–181.
Vinson, J. R. and Chou, T. W. (1975). *Composite Materials and their Use in Structures*. Applied Science Publishers, London.